ISRAEL JOURNAL OF MATHEMATICS, Vol. 54, No. 1, 1986

CONVOLUTION EQUATIONS AND SPACES OF ULTRADIFFERENTIABLE FUNCTIONS

BY

DANIELE C. STRUPPA[†]

Scuola Normale Superiore, Piazza dei Cavalieri, 7. 56100 Pisa, Italy

ABSTRACT

We study spaces of "approximate solutions" to a given convolution equation. In particular we show how their quasianalyticity can be reduced to the study of a suitably constructed Cauchy problem for related spaces. We give a unicity theorem for this problem.

1. In this paper we study some spaces of ultradifferentiable functions and we discuss their quasianalyticity. To this purpose, we act in the framework of the theory of analytically uniform spaces (AU-spaces in the sequel) which, after [1] and [5], seem particularly well suited for this study.

Let $\mathscr{E}_A(L)$ be a (locally convex topological) vector space of C^∞ functions on $\mathbf{R}^n$, which are approximate solutions to a given convolution equation $L * f = 0$, and let $T \subseteq \mathbf{R}^n$ be a linear subvariety. We say that $\mathscr{E}_A(L)$ is T-quasianalytic if there are no (nontrivial) functions f in $\mathscr{E}_A(L)$ which, on T, satisfy $D^\alpha(L^j * f) = 0$ (the reader is referred to section 2 for precise notation and terminology). We show (Theorem 2.3) that it is possible to construct a convolutor μ in $n+1$ variables, and a weight ϕ such that the T-quasianalyticity of $\mathscr{E}_A(L)$ reduces to the uniqueness of the Cauchy problem for the convolutor μ in the space $\mathscr{E}(\phi)$ of C^∞ functions satisfying certain growth conditions induced by ϕ. Finally, in Theorem 3.2, we give explicit conditions on the pair (μ, ϕ), which make $\mathscr{E}(\phi)$ into a uniqueness space for the Cauchy problem for μ

2. Let $\mathscr{E} = \mathscr{E}(\mathbf{R}^n)$ be the space of infinitely differentiable functions equipped with the usual topology of the uniform convergence on the compact subsets of

[†] This paper is dedicated to the author's son, Alessandro.

The author has been partially supported by Ministero P. I., and G.N.S.A.G.A. of Consiglio Nazionale delle Ricerche.

Received April 30, 1984 and in revised form August 1, 1985

$\mathbf{R}^n$, and let $\mathcal{D}$ denote its subspace consisting of all functions with compact support. We recall the definition of the Beurling and the Roumieu ultradifferentiable functions: let $A = \{a_j\}_{j=0}^{+\infty}$ be a convex sequence of positive numbers, i.e. $a_j = \exp(g(j))$, where $g : \mathbf{R}^+ \to \mathbf{R}^+$ is a convex function such that $g(r)/r \to \infty$ when $r \to \infty$, and let $\lambda(z) = \Sigma_j |z|^j / a_j$.

DEFINITION 2.1. The space $\mathcal{E}_{A,R}$ of Roumieu ultradifferentiable functions is the space of all infinitely differentiable functions f on $\mathbf{R}^n$ such that, on every compact set, for every $\varepsilon > 0$ and for every α, there is a positive constant C, depending on ε, α, f and on the compact set, such that

$$|D^\alpha f(x)| \leqq C\varepsilon^{|\alpha|} a_{|\alpha|}.$$

As usual, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multiindex of nonnegative integers, $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $D^\alpha = (\partial^{\alpha_1}/\partial x_1^{\alpha_1}, \ldots, \partial^{\alpha_n}/\partial x_n^{\alpha_n})$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

REMARK 2.1. In Definition 2.1, one can replace "for every $\varepsilon > 0$" by "there exists $\varepsilon > 0$". This leads to nontrivial changes in the theory, especially if ε is allowed to depend on the compact set (see [5]). On the other hand, if ε only depends on f, we have the so-called Beurling spaces $\mathcal{E}_{A,B}$ which can be treated in the same way as the Roumieu spaces (we will not consider these spaces, in the sequel, but everything carries over to them with no essential changes: for this reason we will eliminate the subscripts R and B); the reader interested in the theory of these spaces is referred to [4] and [9].

In [1] and in [5], a more complicated class of function spaces is considered, the spaces $\mathcal{E}_A(L;\mathbf{R}^n) = \mathcal{E}_A(L)$, in connection with some questions of quasianalyticity. Let Q be a linear constant coefficient differential operator: $\mathcal{E}_A(Q)$ is the space of those infinitely differentiable functions f which, together with their derivatives, satisfy, for all $\varepsilon > 0$ and for some $C > 0$,

$$|Q^j f(x)| \leqq C\varepsilon^{|j|} a_{|j|},$$

uniformly on the compact sets of $\mathbf{R}^n$.

More generally, one can replace the differential operator Q by a convolution operator. For f in $\mathcal{E}$ and L in $\mathcal{E}'$, the dual space of $\mathcal{E}$, i.e. the space of all distributions with compact support, one defines the convolution product $L * f$ by using the duality brackets $\langle\ ,\ \rangle$ in the following way:

$$(L * f)(x) = \langle L, f(t - x)\rangle,$$

where L acts on $f(t - x)$ considered as a function of t only. Suppose now

$L = (L_1, \ldots, L_r) \in (\mathscr{E}')^r$: define $L_i * L_k$ by

$$\langle L_i * L_k, f \rangle = \langle L_i, L_k * f \rangle,$$

and, if $j = (j_1, \ldots, j_r)$ is a multiindex,

$$L^j = L_1^{j_1} * \cdots * L_r^{j_r}.$$

DEFINITION 2.2. For A, L as above, $\mathscr{E}_A(L)$ is the space of all infinitely differentiable functions on $\mathbf{R}^n$ such that for all $\varepsilon > 0$, and for all α, there exists $C = C(f, \varepsilon, \alpha)$ such that for all x in $\mathbf{R}^n$ and all j it is

$$|D^\alpha (L^j * f)(x)| \leqq C \varepsilon^{|j|} a_{|j|}.$$

REMARK 2.2. $\mathscr{E}_A(L)$ is a Roumieu space; the corresponding Beurling space can be obtained as in Remark 2.1.

REMARK 2.3. For $L = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, the space $\mathscr{E}_A(L)$ reduces to the space $\mathscr{E}_A$ described earlier: notice that the derivatives D^α have only been introduced with the purpose of making $\mathscr{E}_A(L)$ a Frechet–Montel space, in the natural way induced by the seminorms

$$\|f\|_{\alpha,\varepsilon} = \sum_j (\sup |D^\alpha (L^j * f)(x)|)/a_{|j|} \varepsilon^{|j|}.$$

We will call $\mathscr{E}_A(L)$ the space of ultradifferentiable functions of class (A, L).

DEFINITION 2.3. Denote by $\mathscr{E}'_A(L)$ the strong dual of $\mathscr{E}_A(L)$; we call its elements ultradistributions of class (A, L).

As pointed out in [1], both the spaces $\mathscr{E}_A$ and $\mathscr{E}_A(L)$ are AU-spaces, whose AU-structure is given as follows:

THEOREM 2.1. *The space $\hat{\mathscr{E}}'_A(L)$, of the Fourier transforms of the elements of $\mathscr{E}'_A(L)$, consists of all entire functions $F(z)$ in $\mathbf{C}^n$ which satisfy, for some a, b, $c > 0$,*

$$|F(z)| \leqq a\lambda(c\hat{L}(z))(1 + |z|)^b \exp(b\,|\mathrm{Im}\, z|)$$

where $\hat{L}(z) = \hat{L}_1(z) \cdot \cdots \cdot \hat{L}_r(z)$.

REMARK 2.4. Later on, in this section, we will sketch a proof of this result which is different from the original one given in [1].

We now provide a short description of a number of related spaces which we will use in the sequel. Let ϕ be a continuous, positive, increasing, nonconstant convex function on $\mathbf{R}^n$. $\mathscr{E}(\phi)$ is the reflexive Frechet space of all infinitely differentiable functions on $\mathbf{R}^n$ such that, for all α, for all $\varepsilon > 0$, there exists a constant C such that

$$|D^\alpha f(x)| \leqq Ce^{\phi(\varepsilon x)}.$$

of entire functions; this can be achieved by requiring (see [7]) that ϕ dominates all linear functions: such a function ϕ will be called "admissible". In these hypotheses, if one denotes by ϕ^* the Legendre transform of ϕ, i.e.

$$\phi^*(y) = \max_{x \in \mathbf{R}^n}(x \cdot y - \phi(x)),$$

it is possible to show that ϕ^* is still continuous and convex, and $\hat{\mathscr{E}}'(\phi)$ consists of all entire functions F for which there exist constants $A, a, b > 0$, such that

$$|F(z)| \leqq A(1 + |z|)^a \exp(\phi^*(b \operatorname{Im} z)).$$

This result can now be used to show that $E(\phi)$ is an AU-space, whenever ϕ is admissible.

Further extensions can be obtained considering that $\mathscr{E}(\phi)$ is a space of infinitely differentiable functions which satisfy certain growth conditions with respect to a given convex function ϕ, or, equivalently, with respect to a family Φ of convex functions, where $\Phi = \{\phi(\varepsilon x) : \varepsilon > 0\}$. Hence it is spontaneous to extend the previous example to the case in which Φ is a family of continuous, positive, increasing, nonconstant convex functions ϕ. In this case, under suitable conditions for the family Φ (see [10]), a new space $\mathscr{E}(\Phi)$ can be defined via the seminorms

$$\|f\|_{\alpha,\phi} = \sup\{|D^\alpha f(x)| \exp(-\phi(x))\},$$

and it is now immediate to obtain a description for the space $\hat{\mathscr{E}}'(\Phi)$.

One can finally consider the more general spaces $\mathscr{E}_A(L;\Phi)$, naturally defined by the seminorms

$$\|f\|_{\alpha,\phi,\varepsilon} = \sum_j a_{|j|}\varepsilon^{|j|-1} \sup_x |D^\alpha(L^j * f)(x)| \exp(-\phi(x)).$$

We are now ready to attack the problems of quasianalyticity connected with the spaces which we have introduced. Let $\mathscr{S}$ be any of the above-defined spaces of infinitely differentiable functions (i.e. $\mathscr{S}$ could be $\mathscr{E}$, $\mathscr{E}(\phi)$, $\mathscr{E}_A$, $\mathscr{E}_A(L)$, $\mathscr{E}_A(L;\phi)$, etc.). We say that $\mathscr{S}$ is nonquasianalytic if there exists a nontrivial function f in $\mathscr{S}$ of compact support, or, equivalently, if there exists a nontrivial function f in $\mathscr{S}$, all of whose derivatives vanish at the origin.

If $\mathcal{S} = \mathcal{E}_A$, a necessary and sufficient condition for it to be quasianalytic is given by the well-known theorem of Denjoy–Carleman, [8]:

THEOREM 2.2. *The space $\mathcal{E}_A(\mathbf{R}^n)$ is quasianalytic if, and only if,*

$$\sum a_j^{-1/j} = +\infty.$$

In [5], the space $\mathcal{E}_A(L)$ is considered, for L a partial differential operator with constant coefficients, and a theorem is given to find out if it is possible, for $f \in \mathcal{E}_A(L)$, to vanish (together with all its derivatives) on a noncharacteristic T of L. The reason for the consideration of a noncharacteristic T is the well-known fact that, for the characteristic case, one has to introduce certain growth conditions on the function f in order to obtain some result (this corresponds to saying that uniqueness for the Cauchy problem holds in $\mathcal{E}_A(L;\phi)$, but not necessarily in $\mathcal{E}_A(L)$, for the characteristic case). The noncharacteristic case has been studied, for convolution equations, in [6], and an analog of the famous Holmgren's uniqueness theorem has been obtained. It is our purpose to show how to reduce this problem, for a general convolutor L in $\mathcal{E}'(\mathbf{R}^n)$, to the Cauchy problem studied in [1], [2].

Let us consider L in $\mathcal{E}'(\mathbf{R}^n)$, and let T be a linear subvariety of $\mathbf{R}^n$.

DEFINITION 2.4. The space $\mathcal{E}_A(L)$ is called T-quasianalytic if

$$\{f \text{ in } \mathcal{E}_A(L) : D^\alpha(L^j * f) = 0 \text{ on } T \text{ for all } \alpha, j\} = \{0\}.$$

We want to give conditions on the pair (L, T) which ensure the T-quasianalyticity of the space $\mathcal{E}_A(L)$. For this purpose, consider the immersion of $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$, whose variable w we split as $w = (x, y)$, for x in $\mathbf{R}^n$, and y in $\mathbf{R}$. Let ϕ be an admissible function on $\mathbf{R}$: denote by $\tilde{\mathcal{E}}(\phi)$ the space of all functions $g(x, y)$ in $\mathcal{E}(\mathbf{R}^{n+1})$ whose growth, along the y-coordinate, is bounded by $\exp(\phi(\varepsilon y))$. It is easy to see that $\tilde{\mathcal{E}}(\phi)$ is an AU-space.

Let now μ be an element of $\mathcal{E}'(\mathbf{R}^{n+1})$, and $S_1 \subset S_2$ two linear subvarieties of $\mathbf{R}^{n+1}$.

DEFINITION 2.5. We say that $\tilde{\mathcal{E}}(\phi)$ is a uniqueness space for μ with respect to (S_1, S_2) if the vanishing of a solution f of $\mu * f = 0$ with all its S_2-derivatives on S_1 implies that $f = 0$ on S_2. If $S_1 = S$, $S_2 = \mathbf{R}^{n+1}$, we simply say that $\tilde{\mathcal{E}}(\phi)$ is a uniqueness space for the Cauchy problem for μ on S.

To connect Definitions 2.4 and 2.5, construct a function ϕ on $\mathbf{R}$ by

$$\exp(\phi(y)) = \sum a_j |y|^{(m+1)j}/((m+1)j)!. \tag{1}$$

We make the following requirement on A:

(i) there are C, $m > 0$ such that

$$a_j \leqq C(mj)!, \tag{2}$$

(ii) the function ϕ defined by (1) is admissible, and makes $\tilde{\mathscr{E}}(\phi)$ into a LAU-space (see [5]),

(iii) there is $D > 0$ such that $\lambda(z) \leqq \exp(D|z|)$.

Finally, define the convolutor μ in $\mathscr{E}'(\mathbf{R}^{n+1})$ by

$$\mu * g = (-1)^{m+1}\partial^{m+1} g/\partial y^{m+1} - L * g.$$

We can now prove the

THEOREM 2.3. *Let $L \in \mathscr{E}'(\mathbf{R}^n)$ and $T = \{x \in R^n : x_n = 0\}$. If $\tilde{\mathscr{E}}(\phi)$ is a uniqueness space for the Cauchy problem for μ with respect to $(T, y = 0)$, then $\mathscr{E}_A(L)$ is T-quasianalytic.*

PROOF. For m satisfying (2), the series

$$\sum_j L^j * f(x) y^{(m+1)j}/((m+1)j)!, \qquad x \in \mathbf{R}^n, \quad y \in \mathbf{R}, \tag{3}$$

converges to a function $g(x, y) \in \mathscr{E}$ in the topology of $\mathscr{E}$ itself. Moreover g satisfies the following "heat-like" equation

$$L * g = (-1)^{m+1}\partial^{m+1} g/\partial y^{m+1}, \tag{4}$$

in $\mathscr{E}(\mathbf{R}^{n+1})$. For x in a compact set, one immediately estimates the growth of g by

$$|g(x, y)| \leqq \sum H\varepsilon^{|j|} a_{|j|} |y|^{(m+1)j}/((m+1)j)! = He^{\phi(\varepsilon^{1/(m+1)}y)}.$$

The hypotheses which we have made on A show that g belongs to $\tilde{\mathscr{E}}(\phi)$, and that this is a LAU-space. In such a situation, it is then possible to apply the fundamental principle for "heat-like" convolution equations proved in [1] to the equation (4), thus obtaining a representation of g as

$$g(x, y) = \int_V \partial e^{i(xz + y\zeta)} d\nu(z, \zeta)/k(z, \zeta), \tag{5}$$

where (z, ζ) are the dual variables, in $\mathbf{C}^{n+1}$, of (x, y), (V, ∂) is the multiplicity variety defined (in the (z, ζ)-space) by the equation

$$\hat{L}(z) - (i\zeta)^{m+1} = 0,$$

and where $k(z, \zeta)$ belongs to the AU-structure defining $\tilde{\mathscr{E}}(\phi)$.

In particular, (3) and (5) yield

$$f(x) = g(x,0) = \int_V \partial e^{ixz} d\nu(z,\zeta)/k(z,\zeta). \tag{6}$$

With a standard argument (see [5], theorems 5.26 and 13.1), one deduces from (6) and from (iii) that f belongs to $\mathscr{E}_A(L)$ if and only if g belongs to $(\operatorname{Ker}\mu) \cap \tilde{\mathscr{E}}(\phi)$, for μ defined as above. This concludes the proof. □

REMARK 2.5. This uniqueness result shows that $\mathscr{E}_A(L)$ is T-quasianalytic if $\tilde{\mathscr{E}}(\phi)$ is "quasianalytic" with respect to the convolutor μ, thus shifting the problem to the study of the properties of $\tilde{\mathscr{E}}(\phi)$.

REMARK 2.6. Notice that, even if T is noncharacteristic for L, T may well be characteristic for μ, hence, even in the noncharacteristic case, we have to deal with some kind of growth conditions.

REMARK 2.7. The proof just given can be used to give a new proof of Theorem 2.1. Indeed, once the correspondence between $\mathscr{E}_A(L)$ and the space $(\operatorname{Ker}\mu) \cap \tilde{\mathscr{E}}(\phi)$ is proved, one can use the fundamental principle for convolution equations (we will suppose, for this purpose, that μ be slowly decreasing) to show that $(\operatorname{Ker}\mu) \cap \tilde{\mathscr{E}}(\phi)$ is still (roughly speaking) an AU-space. From this, it is not too difficult to exhibit a specific AU-structure for $\mathscr{E}_A(L)$.

EXAMPLE 2.1. If $n = 1$, $L = d/dx$ and $T = \{0\}$, then Theorem 2.3 gives a classical condition for the quasianalyticity of the space $\mathscr{E}_A$. In this case, in fact, μ is essentially the heat operator, for which a well-known uniqueness result has been proved by Täcklind, [11], where, for the first time, appears the idea of considering growth conditions associated to a given differential equation.

EXAMPLE 2.2. If L is a differential operator, then Theorem 2.2 particularizes to theorem 13.3 of [5].

3. As a natural consequence of the results described in the previous section, one is led to the study of the uniqueness problems in the spaces $\mathscr{E}(\phi)$ for ϕ a suitable function. In this section we will give conditions on the pair (μ, ϕ) which make $\mathscr{E}(\phi)$ a uniqueness space for the convolutor μ. It is well known that, for D a partial differential operator, any f in $\mathscr{E}$ with $Df = 0$ is determined by its Cauchy data on any noncharacteristic hypersurface, say $t = 0$. In [6] this result is extended to convolution equations, introducing a new definition of noncharacteristic vector, which yields

THEOREM 3.1. [6] *Let* $\mu \in \mathscr{E}'(\mathbf{R}^n)$ *and suppose* $\{x_1 = 0\}$ *be noncharacteristic for it. Then every* f *in* $\mathscr{E}(\mathbf{R}^n)$ *with* $\mu * f = 0$ *and* $\mathrm{supp}(f) \subset \{x \in \mathbf{R}^n : x_1 < 0\}$ *is identically zero in* $\mathbf{R}^n$.

Suppose now we look for a result similar to Theorem 3.1 for the space $\mathscr{E}(\phi)$ (from now on we will write $\mathscr{E}(\phi)$ for $\tilde{\mathscr{E}}(\phi)$) when T is not necessarily noncharacteristic for μ. Notice that, for our purposes, a result concerning the most general μ in $\mathscr{E}'(\mathbf{R}^{n+1})$ is not necessary. In fact, because of the construction developed in the proof of Theorem 2.3, it will be sufficient to study a convolutor of the form

$$\mu * f = \partial^{m+1} f / \partial y^{m+1} - L * f, \tag{7}$$

for f in $\mathscr{E}(\mathbf{R}^{n+1})$ and L in $\mathscr{E}'(\mathbf{R}^n)$; this restriction, however, does not seem to be necessary, as we now have a Fundamental Principle for more general convolution equations, [3].

Let us first explain the ideas behind the uniqueness result which we have in mind (these ideas are due to Ehrenpreis, [5], and, suitably modified, have also been utilized by Berenstein and Dostal in [1]). For t a real parameter (we can actually take $0 \leqq t \leqq 1$), we look for a region A in $\mathbf{C}^{n+1}$ (containing the variety of the zeroes of $\hat{\mu}$) and a class $\mathscr{H}$ of functions $H(t, w)$, $w \in \mathbf{C}^{n+1}$, such that, for a given quasianalytic sequence $B = (b_j)$:

(a) $H(t, w)$ is, for t fixed, analytic in w and, for w fixed, belongs to $\mathscr{E}_B$.

(b) For δ any derivative in t, $\delta H(0, w)$ is a finite sum of terms, each of which is a polynomial in s, for $w = (z, s)$, $z \in \mathbf{C}^n$ and $s \in \mathbf{C}$, multiplied by an element of $\hat{\mathscr{E}}'(\phi)$.

(c) The set $\{H(1, w) : H \in \mathscr{H}\}$ is total in $\hat{\mathscr{E}}'(\phi)(T)$.

(d) For any k in an AU-structure for $\mathscr{E}(\phi)$, the functions $H(t, w)/k(w)$ are uniformly bounded in $\mathscr{E}_B$ for w in A.

(e) For any k in an AU-structure for $\mathscr{E}(\phi)$, and any $c > 0$, it is

$$\sup |\partial^j / \partial t^j H(t, w)| / c^{|j|+1} b_{|j|} k(w) \exp(|s|) < +\infty.$$

Once we have constructed such a family $\mathscr{H}$, the proof of lemma 9.27 of [5] can be followed to establish:

LEMMA 3.1. *If a family* $\mathscr{H}$ *as above exists, then* $\mathscr{E}(\phi)$ *is a uniqueness space for the Cauchy problem for* μ *on* T.

REMARK 3.1. In the proof of lemma 9.27, Ehrenpreis uses his Fundamental Principle for partial differential equations. This is the only point in which we differ from his proof, since we will now use the extended version of [1] of the Fundamental Principle for convolution equations.

Let us now consider a convolutor defined as in (7), and suppose that L in $\mathscr{E}'(\mathbf{R}^n)$ satisfies

$$|\hat{L}(z)|^{1/m+1} \leqq A + B|\operatorname{Re} z|^C + \rho(|\operatorname{Im} z|),$$

where $A, B, C > 0$ and ρ is a positive, continuous, strictly increasing function such that, for any $\delta > 0$, there are $\delta', \delta'' > 0$ satisfying

$$\delta\rho(u) \leqq \rho(\delta' u) + \delta''.$$

Notice that the inverse function $\rho_{-1}(u)$ of ρ is well defined. Let us decompose $\phi(x, y)$ as

$$\phi(x, y) = \phi_1(x) + \phi_2(y),$$

with $\phi_2(y) = 0$ for $|y| < 1$ and $\phi_2(y) = +\infty$ for $|y| \geqq 1$, and consider a function $\psi(x)$ such that $\psi^*(|x|) = \phi_1(x)$. Furthermore, we take ψ convex, positive, and such that

$$\lambda(u) := \exp(\psi(\rho_{-1}(u))) \tag{8}$$

is a positive strictly increasing function of $u \geqq 0$, such that $\log \lambda(u)$ is a convex function of $\log u$, with $\lim_\infty u^j/\lambda(u) = 0$. In particular we will assume we can construct (see [1], [5]) a convex sequence $M = \{m_j\}$ such that, for some positive D,

$$\lambda_M(u/2) \leqq D\lambda(u) \qquad \text{and} \qquad \lambda_M(u) \leqq \exp(|u|).$$

Finally we will suppose that, for some $a \geqq C$,

$$|u|^a = O(\psi(u)). \tag{9}$$

We quote, from [1], the following

LEMMA 3.2. *Let $\psi(u)$ be an even function (positive and convex) which satisfies (9). Then the set of all entire functions F such that*

$$F(z) = O(\exp(-C|\operatorname{Re} z|^a + \psi(d \operatorname{Im} z))) \tag{10}$$

is dense in $\hat{\mathscr{E}}'(\phi)$.

We can now state our final result:

THEOREM 3.2. *Let $T \subset \mathbf{R}^{n+1}$ be defined by $T = \{y = 0\}$; then, with the notation as above, a sufficient condition for $\mathscr{E}(\phi)$ to be a uniqueness space for the Cauchy* *problem for μ on T is that*

$$\int_1^{+\infty} \psi(\rho_{-1}(u))u^{-2}du = +\infty. \tag{11}$$

PROOF. Because of Lemma 3.1, we must construct a region A in $\mathbf{C}^{n+1}$, containing the variety $\hat{\mu} = 0$, and a class $\mathcal{H}$ of functions such that, with respect to the quasianalytic class M constructed above (M is quasianalytic because of (11)), properties (a) through (e) are satisfied. Define $\mathcal{H}$ to be the family of all functions $H(t, w)$, for w in $\mathbf{C}^{n+1}$, $t \in [0, 1]$, given by

$$H(t, w) = e^{it\sigma s}F_1(z_1)\cdot \cdots \cdot F_n(z_n),$$

where $\sigma \in \mathbf{R}$, $s \in \mathbf{C}$ and F_i satisfy (10). It is immediate to observe that properties (a), (b), (c) are satisfied by $\mathcal{H}$. Property (d) holds when A is

$$A = \{w = (z, s) : |s| \leqq |\hat{L}(z)|\},$$

and this is a consequence of our assumptions on ρ.

Our proof is then complete if we prove (e). Observe that, for $\sigma = 1$ fixed,

$$\sup|\partial^j/\partial y^j H(y, w)|/m_{|j|}k(z)\exp(|s|) \leqq \sup|s|^j|F(z)|/m_{|j|}k(z)\exp(|s|).$$

Because of (9) this last expression is majored by

$$(\sup|F(z)|/k(z))(\sup|s|^j/m_{|j|}\lambda(s)) \leqq \sup|F(z)|/k(z) < +\infty,$$

the last inequality being a consequence of our choice of the functions F. The proof of Theorem 3.2 is now complete. □

REMARK 3.2. In Theorem 3.2, the subvariety T is supposed to have codimension 1. This restriction, however, is not at all necessary as one can suitably modify the previous results to allow (similarly to what Ehrenpreis did for partial differential equations in [5]) one to deal with higher codimension varieties and with the notion of (S_1, S_2) uniqueness. So, if codim $T = r > 1$, the region A which appears in the proofs of the results above must be taken differently. In particular one sees that, in proving properties (d) and (e), s and z now have dimensions r and $n + 1 - r$, and

$$A = \{w = (z, s) : |s|^{1/m+1} \leqq A + B|\mathrm{Re}\, z|^C + \rho(|\mathrm{Im}\, z|)\}.$$

REFERENCES

1. C. A. Berenstein and M. A. Dostal, *Analytically Uniform Spaces and their Applications to Convolution Equations*, Lecture Notes in Math., No. 256, Springer-Verlag, Berlin, 1972.

2. C. A. Berenstein and J. Lesmes, *The Cauchy problem for convolution operators. Uniqueness*, Michigan Math. J. **26** (1979), 333–349.

3. C. A. Berenstein and B. A. Taylor, *Interpolation problems in* $\mathbf{C}^n$ *with applications to harmonic analysis*, J. Analyse Math. **38** (1980), 188–254.

4. G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1966), 351–407.

5. L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley–Interscience, New York, 1970.

6. S. Hansen, *Uniqueness of the Cauchy problem for convolution equations*, J. Reine Angew. Math. **317** (1980), 1–15.

7. L. Hörmander, *La transformation de Legendre et la théorème de Paley–Wiener*, C. R. Seances Acad. Sci. **240** (1955), 392–395.

8. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc., New York, 1934.

9. C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. Ecole Norm. Sup. 3 ser., **77** (1960), 41–121.

10. B. A. Taylor, *A semi-norm topology for some (DF)-spaces of entire functions*, Duke Math. J. **38** (1971), 379–385.

11. S. Täcklind, *Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique*, Nova Acta R. Soc. Sci. Ups. **10** (1936), 1–56.